Entire and Meromorphic Solutions of Ordinary Differential Equations

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1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

$$\begin{aligned} \dot{x} &= JH'(t,x) \\ x(0) &= x(T) \end{aligned}$$

with $H(t,\cdot)$ a convex function of x, going to $+\infty$ when $\|x\| \to \infty$.

1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian $H(x)$ is autonomous. For the sake of simplicity, we shall also assume that it is C^1.

We shall first consider the question of nontriviality, within the general framework of (A_∞, B_∞)-subquadratic Hamiltonians. In the second subsection, we shall look into the special case when H is $(0, b_\infty)$-subquadratic, and we shall try to derive additional information.

The General Case: Nontriviality. We assume that H is (A_∞, B_∞)-subquadratic at infinity, for some constant symmetric matrices A_∞ and B_∞, with $B_\infty - A_\infty$ positive definite. Set:

$$\gamma := \text{smallest eigenvalue of} \quad B_\infty - A_\infty \tag{1}$$

$$\lambda := \text{largest negative eigenvalue of} \quad J\frac{d}{dt} + A_\infty\ . \tag{2}$$

Theorem 4 tells us that if $\lambda + \gamma < 0$, the boundary-value problem:

$$\begin{aligned} \dot{x} &= JH'(x) \\ x(0) &= x(T) \end{aligned} \tag{3}$$

has at least one solution $\overline{x}$, which is found by minimizing the dual action functional:

$$\psi(u) = \int_o^T \left[\frac{1}{2} \left(\Lambda_o^{-1} u, u \right) + N^*(-u) \right] dt \tag{4}$$

on the range of Λ, which is a subspace $R(\Lambda)_L^2$ with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2} \left(A_\infty x, x \right) \tag{5}$$

is a convex function, and

$$N(x) \le \frac{1}{2} \left(\left(B_\infty - A_\infty \right) x, x \right) + c \quad \forall x \ . \tag{6}$$

Proposition 1. *Assume $H'(0) = 0$ and $H(0) = 0$. Set:*

$$\delta := \liminf_{x \to 0} 2N(x) \left\| x \right\|^{-2} \ . \tag{7}$$

If $\gamma < -\lambda < \delta$, the solution $\overline{u}$ is non-zero:

$$\overline{x}(t) \ne 0 \quad \forall t \ . \tag{8}$$

Proof. Condition (7) means that, for every $\delta' > \delta$, there is some $\varepsilon > 0$ such that

$$\left\| x \right\| \le \varepsilon \Rightarrow N(x) \le \frac{\delta'}{2} \left\| x \right\|^2 \ . \tag{9}$$

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an $\eta > 0$ such that

$$f \left\| x \right\| \le \eta \Rightarrow N^*(y) \le \frac{1}{2\delta'} \left\| y \right\|^2 \ . \tag{10}$$

Fig. 1. This is the caption of the figure

Since u_1 is a smooth function, we will have $\left\| hu_1 \right\|_\infty \le \eta$ for h small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \le \frac{h^2}{2} \frac{1}{\lambda} \left\| u_1 \right\|_2^2 + \frac{h^2}{2} \frac{1}{\delta'} \left\| u_1 \right\|^2 \ . \tag{11}$$

If we choose δ' close enough to δ, the quantity $\left(\frac{1}{\lambda} + \frac{1}{\delta'}\right)$ will be negative, and we end up with

$$\psi(hu_1) < 0 \qquad \text{for} \quad h \neq 0 \quad \text{small} \ . \tag{12}$$

On the other hand, we check directly that $\psi(0) = 0$. This shows that 0 cannot be a minimizer of ψ, not even a local one. So $\overline{u} \neq 0$ and $\overline{u} \neq \Lambda_o^{-1}(0) = 0$. □

Corollary 2. *Assume H is C^2 and (a_∞, b_∞)-subquadratic at infinity. Let $\xi_1, \dots, \xi_N$ be the equilibria, that is, the solutions of $H'(\xi) = 0$. Denote by ω_k the smallest eigenvalue of $H''(\xi_k)$, and set:*

$$\omega := \text{Min}\,\{\omega_1, \dots, \omega_k\} \ . \tag{13}$$

If:

$$\frac{T}{2\pi} b_\infty < -E\left[-\frac{T}{2\pi} a_\infty\right] < \frac{T}{2\pi}\omega \tag{14}$$

then minimization of ψ yields a non-constant T-periodic solution $\overline{x}$.

We recall once more that by the integer part $E[\alpha]$ of $\alpha \in \mathbb{R}$, we mean the $a \in \mathbb{Z}$ such that $a < \alpha \leq a + 1$. For instance, if we take $a_\infty = 0$, Corollary 2 tells us that $\overline{x}$ exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi} \tag{15}$$

or

$$T \in \left(\frac{2\pi}{\omega}, \frac{2\pi}{b_\infty}\right) \ . \tag{16}$$

Proof. The spectrum of Λ is $\frac{2\pi}{T}\mathbb{Z} + a_\infty$. The largest negative eigenvalue λ is given by $\frac{2\pi}{T}k_o + a_\infty$, where

$$\frac{2\pi}{T}k_o + a_\infty < 0 \leq \frac{2\pi}{T}(k_o + 1) + a_\infty \ . \tag{17}$$

Hence:

$$k_o = E\left[-\frac{T}{2\pi} a_\infty\right] \ . \tag{18}$$

The condition $\gamma < -\lambda < \delta$ now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T}k_o - a_\infty < \omega - a_\infty \tag{19}$$

which is precisely condition (14). □

Lemma 3. *Assume that H is C^2 on $\mathbb{R}^{2n} \setminus \{0\}$ and that $H''(x)$ is non-degenerate for any $x \neq 0$. Then any local minimizer $\widetilde{x}$ of ψ has minimal period T.*

Proof. We know that $\widetilde{x}$, or $\widetilde{x} + \xi$ for some constant $\xi \in \mathbb{R}^{2n}$, is a T-periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x) \ . \tag{20}$$

There is no loss of generality in taking $\xi = 0$. So $\psi(x) \geq \psi(\widetilde{x})$ for all $\widetilde{x}$ in some neighbourhood of x in $W^{1,2}\left(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}\right)$.

But this index is precisely the index $i_T(\widetilde{x})$ of the T-periodic solution $\widetilde{x}$ over the interval $(0, T)$, as defined in Sect. 2.6. So

$$i_T(\widetilde{x}) = 0 \ . \tag{21}$$

Now if $\widetilde{x}$ has a lower period, T/k say, we would have, by Corollary 31:

$$i_T(\widetilde{x}) = i_{kT/k}(\widetilde{x}) \geq k i_{T/k}(\widetilde{x}) + k - 1 \geq k - 1 \geq 1 \ . \tag{22}$$

This would contradict (21), and thus cannot happen. □

Notes and Comments. The results in this section are a refined version of [1]; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family x_T, $T \in \left(2\pi\omega^{-1}, 2\pi b_\infty^{-1}\right)$ of periodic solutions, $x_T(0) = x_T(T)$, with x_T going away to infinity when $T \to 2\pi\omega^{-1}$, which is the period of the linearized system at 0.

Table 1. This is the example table taken out of *The TEXbook,* p. 246

Year	World population
8000 B.C.	5,000,000
50 A.D.	200,000,000
1650 A.D.	500,000,000
1945 A.D.	2,300,000,000
1980 A.D.	4,400,000,000

Theorem 4 (Ghoussoub-Preiss). *Assume $H(t,x)$ is $(0,\varepsilon)$-subquadratic at infinity for all $\varepsilon > 0$, and T-periodic in t*

$$H(t, \cdot) \qquad \text{is convex} \ \ \forall t \tag{23}$$

$$H(\cdot, x) \qquad \text{is} \ \ T\text{−periodic} \ \ \forall x \tag{24}$$

$$H(t,x) \geq n\left(\|x\|\right) \qquad \text{with} \quad n(s)s^{-1} \to \infty \quad \text{as} \quad s \to \infty \tag{25}$$

$$\forall \varepsilon > 0 \ , \quad \exists c \ : \ H(t,x) \leq \frac{\varepsilon}{2} \left\|x\right\|^2 + c \ . \tag{26}$$

Assume also that H is C^2, and $H''(t,x)$ is positive definite everywhere. Then there is a sequence x_k, $k \in \mathbb{N}$, of kT-periodic solutions of the system

$$\dot{x} = JH'(t,x) \tag{27}$$

such that, for every $k \in \mathbb{N}$, there is some $p_o \in \mathbb{N}$ with:

$$p \geq p_o \Rightarrow x_{pk} \neq x_k \ . \tag{28}$$

*Example 5 (*External forcing*).* Consider the system:

$$\dot{x} = JH'(x) + f(t) \tag{29}$$

where the Hamiltonian H is $(0, b_\infty)$-subquadratic, and the forcing term is a distribution on the circle:

$$f = \frac{d}{dt} F + f_o \qquad \text{with} \quad F \in L^2 \left(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}\right) \ , \tag{30}$$

where $f_o := T^{-1} \int_o^T f(t)dt$. For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi \ , \tag{31}$$

where δ_k is the Dirac mass at $t = k$ and $\xi \in \mathbb{R}^{2n}$ is a constant, fits the prescription. This means that the system $\dot{x} = JH'(x)$ is being excited by a series of identical shocks at interval T.

Definition 6. Let $A_\infty(t)$ and $B_\infty(t)$ be symmetric operators in $\mathbb{R}^{2n}$, depending continuously on $t \in [0,T]$, such that $A_\infty(t) \leq B_\infty(t)$ for all t.

A Borelian function $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is called (A_∞, B_∞)-*subquadratic at infinity* if there exists a function $N(t,x)$ such that:

$$H(t,x) = \frac{1}{2} \left(A_\infty(t)x, x\right) + N(t,x) \tag{32}$$

$$\forall t \ , \quad N(t,x) \qquad \text{is convex with respect to} \quad x \tag{33}$$

$$N(t,x) \geq n\left(\|x\|\right) \qquad \text{with} \quad n(s)s^{-1} \to +\infty \quad \text{as} \quad s \to +\infty \tag{34}$$

$$\exists c \in \mathbb{R} \ : \quad H(t,x) \leq \frac{1}{2} \left(B_\infty(t)x, x\right) + c \quad \forall x \ . \tag{35}$$

If $A_\infty(t) = a_\infty I$ and $B_\infty(t) = b_\infty I$, with $a_\infty \leq b_\infty \in \mathbb{R}$, we shall say that H is (a_∞, b_∞)-subquadratic at infinity. As an example, the function $\|x\|^\alpha$, with $1 \leq \alpha < 2$, is $(0, \varepsilon)$-subquadratic at infinity for every $\varepsilon > 0$. Similarly, the Hamiltonian

$$H(t, x) = \frac{1}{2} k \|k\|^2 + \|x\|^\alpha \tag{36}$$

is $(k, k + \varepsilon)$-subquadratic for every $\varepsilon > 0$. Note that, if $k < 0$, it is not convex.

Notes and Comments. The first results on subharmonics were obtained by Rabinowitz in [5], who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on H'. Again the duality approach enabled Clarke and Ekeland in [2] to treat the same problem in the convex-subquadratic case, with growth conditions on H only.

Recently, Michalek and Tarantello (see [3] and [4]) have obtained lower bound on the number of subharmonics of period kT, based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

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Finite Element Approximations and the Dirichlet Problem for Surfaces of Prescribed Mean Curvature*

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1 *H*-Harmonic Maps

The numerical solution of the classical H-Plateau Problem consists of approximating disc-like surfaces with prescribed boundary curve and prescribed mean curvature H. For a detailed discussion of the algorithms and theory see [6] for the case of zero mean curvature, and [7] for the constant mean curvature case. In this paper we consider the associated H-Dirichlet problem.

Estimates for finite element approximations to solutions of general nonlinear elliptic systems are obtained in [4], using a continuity method involving L^∞ estimates for the discrete problem. Here we give a much shorter proof of the H^1 estimate, avoiding the need for L^∞ estimates and only assuming the discrete and smooth data are close in the H^1 sense. Our techniques apply to a wide class of nonlinear systems. We treat the case of a non-polygonal and non-convex boundary and give the explicit dependence on the non-degeneracy constant of the smooth solution being approximated. The arguments are prototypes of those used in [7] for treating the more difficult case of the (free boundary) H-Plateau Problem. The main tool for avoiding L^∞ norms in the present "borderline" case is the isoperimetric inequality due to Rado, see Remark 10.

Throughout, $\Omega\,(\subset \mathbb{R}^2)$ is a bounded domain with C^2 boundary. Function spaces will consist of functions defined over Ω with values in $\mathbb{R}^3$ unless otherwise clear from context. Constants will depend on Ω and other quantities as indicated.

By $|\cdot|_{H^1}$ is meant the H^1 seminorm, and by $\|\cdot\|_{H^1}$ the full norm. Note that by Poincaré's inequality, $|\cdot|_{H^1(\Omega)}$ is a norm on $H^1_0(\Omega)$.

For vectors $a, b, c \in \mathbb{R}^3$, the *triple product* is defined by

$$[a,b,c] = a\cdot b\times c\,.$$

This is invariant under cyclic permutations of a, b and c, and antisymmetric with respect to interchanging any two. It is the volume of the parallelopiped spanned by a, b and c.

* This work was partially supported by the Australian Research Council.

Definition 1. Suppose H is a real number. A function $u \in H^2(\Omega;\mathbb{R}^3)$ is *H-harmonic* with boundary data $u^0 \in H^2(\Omega;\mathbb{R}^3)$ if

$$\Delta u = 2H u_x \times u_y \quad \text{a.e. in } \Omega \tag{1}$$

$$u = u^0 \quad \text{on } \partial\Omega. \tag{2}$$

Example 2. Let D be the closed unit disc in $\mathbb{R}^2$. Let

$$u^0(x,y) = (x,y,0) : \ D \to \mathbb{R}^3$$

with $0 < H < 1$. There are two solutions of (1) and (2) obtained by mapping the unit disc D conformally, i.e. stereographically projecting from a suitable point, onto the *lower* spherical caps obtained from each of the two spheres of radius $1/H$ (mean curvature H) which contain the image of $u^0|_{\partial\Omega}$. These solutions are called *small* or *large* depending on whether their images do not, or do, contain a hemisphere. (We use this example for test computations, see Tables 1 and 2.) If $-1 < H < 0$ then one similarly obtains two solutions from the upper spherical caps. If $H = 0$ then there is exactly one solution, the map $u(x,y) = (x,y,0) : D \to \mathbb{R}^3$. If $H = 1$ then one obtains a solution by mapping onto the lower hemisphere of a sphere of radius 1, and onto the upper hemisphere if $H = -1$.

Equation (1) is the Euler-Lagrange system associated to the *H-Dirichlet integral*

$$D_H(u) = D_H(u;\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 + 2HV(u), \tag{3}$$

where

$$V(u) = V(u;\Omega) := \frac{1}{3}\int_\Omega [u, u_x, u_y] \tag{4}$$

can be thought of as the signed volume of the cone over the origin obtained from the image of u. In fact, direct computation and integration by parts easily gives

$$\begin{aligned} \langle {D_H}'(u), \varphi\rangle = \langle {D_H}'(u;\Omega), \varphi\rangle &:= \frac{d}{dt}\Big|_{t=0} D_H(u + t\varphi) \\ &= \int_\Omega \nabla u \nabla\varphi + 2H\int_\Omega [\varphi, u_x, u_y] \end{aligned} \tag{5}$$

for $u \in C^2(\bar{\Omega};\mathbb{R}^3)$ and $\varphi \in C_0^2(\bar{\Omega};\mathbb{R}^3)$, and hence for $u \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ and $\varphi \in H_0^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ by a limit argument, for example see [10, Remark III.1.1]. If $u \in H^1 \cap L^\infty$ is stationary for D_H, i.e.

$$\int_\Omega \nabla u \nabla\varphi + 2H\int_\Omega [\varphi, u_x, u_y] = 0 \tag{6}$$

for all $\varphi \in H_0^1 \cap L^\infty$, then u is said to be a *weak* solution of (1).

Example 2 is fairly typical. Arguing heuristically, the energy functional $D_H(u)$ is cubic in u and thus one expects (generically) either two or no stationary points. In the former case one expects the "smaller" solution to be a local minimum and the "larger" solution to be unstable.

Indeed, one has the following result due to the combined work of Heinz, Werner, Hildebrandt, Jäger, Wente, Brezis–Coron, Struwe and Steffen. For detailed references see Struwe [10,11].

Theorem 3. *Assume* $u^0 \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ *and* $H \in \mathbb{R}$ *satisfy*

$$\|u^0\|_{L^\infty}|H| \leq 1.$$

Then there exists $\underline{u} \in u^0 + H_0^1 \cap L^\infty$ *such that*

$$D_H(\underline{u}) = \min\left\{ D_H(v) : v \in u^0 + H_0^1,\ \|v\|_{L^\infty}|H| \leq 1 \right\}.$$

Moreover,

$$\|\underline{u}\|_{L^\infty} \leq \|u^0\|_{L^\infty} \qquad (*)$$

and $\underline{u}$ *is a weak solution to* (1) *and* (2).

If furthermore

$$\|u^0\|_{L^\infty}|H| < 1 \qquad (**)$$

then $\underline{u}$ *is the* unique *local minimum of* D_H *in* $u^0 + H_0^1 \cap L^\infty$. *Moreover,* $\underline{u}$ *is the unique weak solution of* (1) *and* (2) *which satisfies* $(*)$. *The function* $\underline{u}$ *is called the* small *solution of* (1) *and* (2).

Under the same assumption $(**)$ *if* $H \neq 0$ *and* u^0 *is not constant, there is also a second weak solution* $\bar{u}$ *to* (1) *and* (2) *which satisfies*

$$\|\bar{u}\|_{L^\infty} > \|u^0\|_{L^\infty}.$$

Any such solution is called a large *solution to* (1) *and* (2).

If $u^0 \in H^2(\Omega, \mathbb{R}^3)$ *then any weak solution to* (1) *and* (2) *belongs to* $H^2(\Omega, \mathbb{R}^3)$.

Remark 4.

1. The large solution need not be unique, although one would expect that this is the generic situation. An example of Wente, [11, Example IV.3.7], gives a continuum of solutions for Ω the unit disc and boundary data $u^0(x, y) = (x, 0, 0)$. See Fig. 1 for the image of the trivial small solution and of one of the large solutions on a relatively coarse grid. Rotation of $u(\Omega)$ around the u_1-axis gives a continuum of solutions.
2. The existence of a large solution is obtained by a mountain pass type argument, see [2] and [11, Theorem III.4.8]

Fig. 1. Wente's example (discrete approximations); small and one of a continuum of large solutions

Remark 5 (Nondegeneracy). We will be interested in approximating functions $u \in H^2(\Omega;\mathbb{R}^3)$ which are H-harmonic and nondegenerate in the sense that the second variation $D_H{}''(u)$ has no zero eigenvalues. This is always true for small solutions, see [11, Lemma III.4.7].

More precisely, for $u \in H^2(\Omega;\mathbb{R}^3)$ and $\varphi, \psi \in H_0^2(\Omega;\mathbb{R}^3)$ one first easily checks by direct computation and integration by parts that

$$
\begin{aligned}
D_H{}''(u)(\varphi,\psi) = D_H{}''(u;\Omega)(\varphi,\psi) &:= \left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s=t=0} D_H(u+t\varphi+s\psi) \\
&= \int_\Omega \nabla\varphi\nabla\psi + 2H\int_\Omega [u,\varphi_x,\psi_y] + [u,\psi_x,\varphi_y] \qquad (7)\\
&= \int_\Omega \nabla\varphi\nabla\psi + 2H\int_\Omega [\psi,u_x,\varphi_y] + [\psi,\varphi_x,u_y], \qquad (8)
\end{aligned}
$$

see [10, Remark III.1.1] and the paragraph following (17). From (8) $D_H{}''(u)$ extends to a bounded symmetric bilinear functional on H_0^1, since

$$
\int_\Omega |\nabla u|\,|\varphi|\,|\nabla\varphi| \le \|\nabla u\|_{L^4}\|\varphi\|_{L^4}\|\nabla\varphi\|_{L^2} \le c\|u\|_{H^2}\|\nabla\varphi\|_{L^2}^2
$$

It follows that the inner product $|\cdot|_{H^1}$ induces a bounded self-adjoint linear operator $\nabla^2 D_H(u) : H_0^1 \to H_0^1$. The eigenvalues of $\nabla^2 D_H(u)$ are real, bounded below by $-\lambda_0$ (say) and have no accumulation point. Moreover, $\lambda_0 = \lambda_0(\Omega, H, \|u\|_{H^2})$, as follows from using (8) to estimate the Raleigh-Ritz quotient.

The *nondegeneracy constant* λ of $D_H{}''(u)$ is defined by

$$
\lambda = \min\left\{\, |\gamma| : \gamma \text{ is an eigenvalue of } \nabla^2 D_H(u) \,\right\}
$$

Then $\nabla^2 D_H(u)$ is one-one and onto iff $\lambda > 0$. Let

$$
\varphi = \varphi^+ + \varphi^- \qquad (9)
$$

denote the $|\cdot|_{H^1}$ orthogonal decomposition of $\varphi \in H_0^1$ into members of the positive and negative spaces H^+ and H^- corresponding to the eigenvalues and eigenfunctions of $\nabla^2 D_H(u)$. Then

$$
D_H{}''(u)(\varphi, \varphi^+ - \varphi^-) \ge \lambda|\varphi|_{H^1}^2 \qquad (10)
$$

for all such φ and λ is the largest such real.

From the eigenfunction equation, c.f. (8), together with the estimate for λ_0, one obtains $\varphi^- \in H^2$ and

$$\|\varphi^-\|_{H^2} \leq \nu|\varphi^-|_{H^1} \tag{11}$$

where $\nu = \nu(\Omega, H, \|u\|_{H^2}, d)$ with d the dimension of H^-.

2 Discrete H-Harmonic Maps

For $h > 0$ let $\mathcal{T}_h$ be a triangulation of Ω by triangles T whose side lengths are bounded above by ch for some c independent of h and whose interior angles are bounded away from zero uniformly and independently of h. The intersection of any two different triangles is either empty, a common vertex, or a common edge.

Let

$$\Omega_h = \bigcup_{T \in \mathcal{T}_h} T.$$

Let

$$\begin{aligned} X_h &= \left\{ u_h \in C^0(\Omega_h; \mathbb{R}^3) : u_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h \right\}, \\ X_{h0} &= \left\{ \varphi_h \in X_h : \varphi_h|_{\partial\Omega_h} = 0 \right\}, \end{aligned}$$

where $\mathbb{P}_1(T)$ is the set of polynomials over T of degree at most one.

For some $\delta > 0$ and all sufficiently small h,

$$\Omega' := \{ \mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \Omega) < \delta \} \supset \Omega_h \cup \Omega.$$

If $u \in H^2(\Omega)$ then by the C^2 regularity of Ω there exists an extension of u to Ω', also denoted by u, such that

$$\|u\|_{H^2(\Omega')} \leq c\|u\|_{H^2(\Omega)}. \tag{12}$$

Definition 6. The *discrete H-Dirichlet integral* is defined by

$$D_H(u_h; \Omega_h) = \frac{1}{2}\int_{\Omega_h} |\nabla u_h|^2 + 2HV(u_h; \Omega_h)$$

for $u_h \in X_h$.

It follows from (5) and (7) with Ω replaced by Ω_h and a limit argument, or by direct computation and noting that boundary integrals on internal edges cancel, that

$$\langle D'_H(u_h; \Omega_h), \varphi_h \rangle = \int_{\Omega_h} \nabla u_h \nabla \varphi_h + 2H \int_{\Omega_h} [\varphi_h, u_{hx}, u_{hy}], \tag{13}$$

$$D_H{}''(u_h; \Omega_h)(\varphi_h, \psi_h) = \int_{\Omega_h} \nabla \varphi_h \nabla \psi_h + 2H \int_{\Omega_h} [u_h, \varphi_{hx}, \psi_{hy}] + [u_h, \psi_{hx}, \varphi_{hy}], \tag{14}$$

for $u_h \in X_h$ and $\varphi_h, \psi_h \in X_{h0}$.

Motivated by (6), one has

Definition 7. A function $u_h \in X_h$ is *discrete H-harmonic* if

$$\int_{\Omega_h} \nabla u_h \nabla \varphi_h + 2H \int_{\Omega_h} [u_{hx}, u_{hy}, \varphi_h] = 0, \tag{15}$$

for all $\varphi_h \in X_{h0}$.

We will prove the following.

Theorem 8. *Let $u \in H^2(\Omega;\mathbb{R}^3)$ be H-harmonic and $u = u^0$ on $\partial\Omega$ where $u^0 \in H^2(\Omega;\mathbb{R}^3)$. Assume u is nondegenerate with nondegeneracy constant λ.*

Let $u_h^0 \in X_h$ and assume $\|u^0 - u_h^0\|_{H^1(\Omega_h)} \le \alpha h$.

Then there exist constants $h_0 = h_0(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha, \Omega, d, H, \lambda)$, $\varepsilon_0 = \varepsilon_0(H, \lambda)$, and $c_0 = c_0(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha, H)$ such that if $0 < h \le h_0$ then:

1. *There exists a* unique *discrete H-harmonic function u_h such that $u_h = u_h^0$ on $\partial\Omega_h$ and*
$$\|u - u_h\|_{H^1(\Omega_h)} \le \varepsilon_0;$$
2. *Moreover,*
$$\|u - u_h\|_{H^1(\Omega_h)} \le c_0 \lambda^{-1} h.$$

3 Proof of Main Theorem

With u, u^0 and u_h^0 as in the main theorem *define*

$$J_h u = u_h^0 + I_h(u - u^0) \in u_h^0 + X_{h0}, \tag{16}$$

where I_h is the standard nodal interpolation operator.

The proof of the main theorem will use the following quantitative version of the Inverse Function Theorem with $\mathcal{X} = u_h^0 + X_{h0}$, $X = X_{h0}$, $Y = X_{h0}^*$ (the dual space of X_{h0}), $x_0 = J_h u$, $f = D'_H(\cdot\,;\Omega_h)$. The proof of the lemma follows from that in [1] pp 113–114.

Lemma 9. *Let $\mathcal{X}$ be an affine Banach space with Banach space X as tangent space, and let Y be a Banach space. Suppose $x_0 \in \mathcal{X}$ and $f \in C^1(\mathcal{X}, Y)$. Assume there are positive constants α, β, δ and ε such that*

$$\|f(x_0)\|_Y \le \delta,$$
$$\|f'(x_0)^{-1}\|_{L(Y,X)} \le \alpha^{-1},$$
$$\|f'(x) - f'(x_0)\|_{L(X,Y)} \le \beta \quad \text{for all } x \in \bar{B}_\varepsilon(x_0),$$

where

$$\beta < \alpha, \quad \delta \le (\alpha - \beta)\varepsilon.$$

Then there exists a unique $x_* \in \bar{B}_\varepsilon(x_0)$ such that $f(x_*) = 0$.

Remark 10 (The Volume Functional). A fundamental result due to Wente [12] is that for any $u^0 \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ the functional V (and hence D_H) extends to an analytic functional on the affine space $u^0 + H^1_0(\Omega;\mathbb{R}^3)$. This is perhaps surprising, since from (5) and (7) one might expect bounds for the relevant integrals to also involve $\|\varphi\|_{L^\infty}$ and $\|u\|_{L^\infty}$ respectively.

More generally, one has the following.

For $u, v, w \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ *define* the trilinear functional

$$V(u,v,w) = V(u,v,w;\Omega) = \frac{1}{6}\int_\Omega [u, v_x, w_y] + [u, w_x, v_y]. \tag{17}$$

Note that $V(u) = V(u,u,u)$.

Assume now that *at least one* of u, v, w also belongs to $H^1_0(\Omega;\mathbb{R}^3)$. Then V is invariant under cyclic permutations of its arguments, as follows from integration by parts in the C^2 case and in general by a limit argument, see [10, Remark III.1.1.iii]. Since V is invariant under permutation of its first two arguments, it then follows it is invariant under *any* permutation of its arguments. Moreover under the same assumptions, from an argument similar to that in [10, proof of Theorem III.2.3] which uses an isoperimetric inequality due to Radó [8], one also has

$$V(u,v,w) \le c|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}. \tag{18}$$

Similar remarks and estimates apply if Ω is everywhere replaced by Ω_h.

Assume now $u, v \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ and $\varphi \in H^1_0 \cap L^\infty(\Omega;\mathbb{R}^3)$. It follows that

$$\langle V'(u), \varphi\rangle = 3V(u,u,\varphi) \le c|u|^2_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}, \tag{19}$$

$$V''(u)(v,\varphi) = 6V(u,v,\varphi) \le c|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}, \tag{20}$$

$$V'''(\cdot)(u,v,\varphi) = 6V(u,v,\varphi) \le c|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)}. \tag{21}$$

(In particular, if $u \in H^1 \cap L^\infty(\Omega;\mathbb{R}^3)$ these estimates allow one to define the integrals in (5) and (7) for arbitrary $\varphi, \psi \in H^1_0(\Omega;\mathbb{R}^3)$.) Similar results also hold if Ω is replaced by Ω_h.

For the remainder of this section, u is as in the Main Theorem. Extend u to Ω' as in (12) and restrict to Ω_h as necessary. Both the extension and restriction will also be denoted by u.

Lemma 11.

$$\|u - J_h u\|_{H^1(\Omega_h)} \le c_1 h$$

where $c_1 = c_1(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha)$.

Proof.

$$\begin{aligned}\|u - J_h u\|_{H^1(\Omega\cap\Omega_h)} &= \left\|(u^0 - u^0_h) - (I_h(u-u^0) - (u-u^0))\right\|_{H^1(\Omega\cap\Omega_h)}\\ &\le \|u^0 - u^0_h\|_{H^1(\Omega\cap\Omega_h)} + ch|u-u^0|_{H^2(\Omega)}\\ &\le ch,\end{aligned}$$

where $c = c(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha)$. Since

$$\|u\|_{H^1(\Omega_h\setminus\Omega)} \le ch\|u\|_{H^2(\Omega_h\setminus\Omega)} \le ch\|u\|_{H^2(\Omega)}$$

by elementary estimates and (12), the result follows.

Lemma 12. *If* $\varphi_h \in X_{h0}$ *then*

$$|\langle D'_H(J_h u; \Omega_h), \varphi_h\rangle| \le c_2 h|\varphi_h|_{H^1(\Omega_h)},$$

where $c_2 = c_2(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, H)$.

Proof.

$$\begin{aligned}
&\langle D'_H(J_h u; \Omega_h), \varphi_h\rangle \\
&\quad = \big(\langle D'_H(J_h u; \Omega_h), \varphi_h\rangle - \langle D'_H(u; \Omega_h), \varphi_h\rangle\big) + \langle D'_H(u; \Omega_h), \varphi_h\rangle \\
&\quad =: A + B
\end{aligned}$$

From the Taylor series expansion for $V'(\cdot\,; \Omega_h)$ and Remark 10

$$\begin{aligned}
|A| &= \left|\int_{\Omega_h} \nabla(J_h u - u)\nabla\varphi_h + 2H\langle V'(J_h u; \Omega_h), \varphi_h\rangle - \langle V'(u; \Omega_h), \varphi_h\rangle\right| \\
&\le |J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} + 2|H||V''(u; \Omega_h)(J_h u - u, \varphi_h)| \\
&\quad + |H||V'''(u; \Omega_h)(J_h u - u, J_h u - u, \varphi_h)| \\
&\le |J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} \\
&\quad + c|H||u|_{H^1(\Omega_h)}|J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} \\
&\quad + c|H||J_h u - u|^2_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)} \\
&\le ch|\varphi_h|_{H^1(\Omega_h)},
\end{aligned}$$

from Lemma 11 and (12), where $c = c(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, H)$.

Also,

$$\begin{aligned}
B &= \left|\int_{\Omega_h} \nabla u\nabla\varphi_h + 2H\int_{\Omega_h}[\varphi_h, u_x, u_y]\right| = \left|\int_{\Omega_h}(-\Delta u + 2Hu_x \times u_y)\cdot\varphi_h\right| \\
&= \left|\int_{\Omega_h\setminus\Omega}(-\Delta u + 2Hu_x \times u_y)\cdot\varphi_h\right| \le c\|\varphi_h\|_{L^2(\Omega_h\setminus\Omega)} \le ch\|\varphi_h\|_{H^1(\Omega_h)}
\end{aligned}$$

where $c = c(\|u\|_{H^2}, H)$, as follows from (12), a Sobolev imbedding theorem, and elementary calculus.

The required result follows.

Remark 13 (A "Discrete Eigenspace" Decomposition). If $\varphi_h \in X_{h0}$ let φ_h also denote the zero extension to $\Omega \cup \Omega_h$. Note that $\varphi_h \notin H_0^1(\Omega)$ unless Ω is convex. For this reason *define* $P : X_{h0} \to H_0^1(\Omega)$ to be the $|\cdot|_{H^1}$ projection, i.e.

$$\int_\Omega \nabla(P\varphi_h)\nabla\varphi = \int_\Omega \nabla\varphi_h\nabla\varphi$$

for all $\varphi \in H_0^1(\Omega)$.

One has

$$|\varphi_h|_{H^1(\Omega_h\setminus\Omega)} \le ch^{1/2}|\varphi_h|_{H^1(\Omega_h)} \tag{22}$$

$$|P\varphi_h|_{H^1(\Omega)} \le |\varphi_h|_{H^1(\Omega_h)} \tag{23}$$

$$|\varphi_h - P\varphi_h|_{H^1(\Omega)} \le ch^{1/2}|\varphi_h|_{H^1(\Omega_h)} \tag{24}$$

To see (22) note that $\nabla\varphi_h$ is constant on any triangle $T \in \mathcal{T}_h$ and that $|T \cap (\Omega_h \setminus \Omega)| \le ch|T|$. Inequality (23) is immediate, since P is just $|\cdot|_{H^1}$ orthogonal projection onto $H_0^1(\Omega)$. For (24) first note that $|\varphi_h - P\varphi_h|_{H^1(\Omega)} \le |\varphi_h - \varphi|_{H^1(\Omega)}$ for any $\varphi \in H_0^1(\Omega)$, by orthogonality. Now choose φ by suitably deforming φ_h in a boundary strip.

Let $(P\varphi_h)^+, (P\varphi_h)^- \in H_0^1(\Omega)$ be the components of $P\varphi_h$ as in (9). Note that $(P\varphi_h)^-$ is smooth, and in particular

$$\|(P\varphi_h)^-\|_{H^2(\Omega)} \le \nu|\varphi_h|_{H^1(\Omega_h)} \tag{25}$$

since

$$\|(P\varphi_h)^-\|_{H^2(\Omega)} \le \nu|(P\varphi_h)^-|_{H^1(\Omega)} \le \nu|P\varphi_h|_{H^1(\Omega)} \le \nu|\varphi_h|_{H^1(\Omega_h)}$$

from (11), (9) and the $|\cdot|_{H^1}$-orthogonality of $(P\varphi_h)^+$ and $(P\varphi_h)^-$, and (23).

Define a discrete analogue of (9) by

$$\begin{gathered}\varphi_h^{(-)} = I_h(P\varphi_h)^- \in X_{h0}, \quad \varphi_h^{(+)} = \varphi_h - \varphi_h^{(-)},\\ \varphi_h = \varphi_h^{(+)} + \varphi_h^{(-)}.\end{gathered} \tag{26}$$

Taking the zero extension of $(P\varphi_h)^-$ and $(P\varphi_h)^+$ to Ω_h, and of $\varphi^{(-)}$ and $\varphi^{(+)}$ to Ω, we claim

$$\begin{aligned}|(P\varphi_h)^- - \varphi_h^{(-)}|_{H^1(\Omega\cup\Omega_h)} &\le ch|\varphi_h|_{H^1(\Omega_h)},\\ |(P\varphi_h)^+ - \varphi_h^{(+)}|_{H^1(\Omega\cup\Omega_h)} &\le ch^{1/2}|\varphi_h|_{H^1(\Omega_h)},\end{aligned} \tag{27}$$

where $c = c(\nu)$

Proof of claim.

$$|(P\varphi_h)^- - \varphi_h^{(-)}|_{H^1(\Omega_h)} \le ch|(P\varphi_h)^-|_{H^2(\Omega)} \le ch\nu|\varphi_h|_{H^1(\Omega_h)}$$

from (25). Also

$$|(P\varphi_h)^-|_{H^1(\Omega\setminus\Omega_h)} \le ch|(P\varphi_h)^-|_{H^2(\Omega)} \le ch\nu|\varphi_h|_{H^1(\Omega_h)}.$$

This gives the first result.

For the second,

$$\begin{aligned}&|(P\varphi_h)^+ - \varphi_h^{(+)}|_{H^1(\Omega_h)}\\ &\quad\le |P\varphi_h - \varphi_h|_{H^1(\Omega_h)} + |(P\varphi_h)^- - \varphi_h^{(-)}|_{H^1(\Omega_h)} \le c(h^{1/2} + h\nu)|\varphi_h|_{H^1(\Omega_h)}\end{aligned}$$

from the first result and (24). On $\Omega \setminus \Omega_h$, $\varphi_h = \varphi_h^{(+)} = 0$ and so the required estimate now follows from (24). □

We also have

$$\begin{aligned}|\varphi_h^{(-)}|_{H^1(\Omega_h)} &\le (1 + ch)|\varphi_h|_{H^1(\Omega_h)}\\ |\varphi_h^{(+)}|_{H^1(\Omega_h)} &\le (1 + ch^{1/2})|\varphi_h|_{H^1(\Omega_h)}\end{aligned} \tag{28}$$

from (27), the orthogonal decomposition $P\varphi_h = (P\varphi_h)^- + (P\varphi_h)^+$ and (23).

Thus (26) is an “almost orthogonal” decomposition for small h.

Lemma 14. *If $\varphi_h \in X_{h0}$ then*

$$D_H''(J_h u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) \ge \frac{3\lambda}{4}|\varphi_h|^2_{H^1(\Omega_h)}$$

provided $h \le h_1$ where $h_1 = h_1(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \Omega, d, H, \lambda)$.

Proof. Since $V(\cdot\,; \Omega_h)$ is cubic, from (14)

$$\begin{aligned}&D_H''(J_h u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)})\\ &\quad= {D_H}''(u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) + 2HV'''(u; \Omega_h)(J_h u - u, \varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}).\end{aligned}$$

But

$$\begin{aligned}&\left|2HV'''(u; \Omega_h)(J_h u - u, \varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)})\right|\\ &\qquad\le c|J_h u - u|_{H^1(\Omega_h)}|\varphi_h|_{H^1(\Omega_h)}|\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h)} \le ch|\varphi_h|^2_{H^1(\Omega_h)}\end{aligned}$$

where $c = c(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \nu, H)$, from Remark 10, also Lemma 11 and (28).

Now

$${D_H}''(u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) = {D_H}''(u; \Omega)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) + E_1$$

where

$$|E_1| \le c(1 + \|u\|_{L^\infty})|\varphi_h|_{H^1(\Omega_h\setminus\Omega)}|\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h\setminus\Omega)} \le ch|\varphi_h|^2_{H^1(\Omega_h)}$$

with $c = c(\|u\|_{H^2(\Omega)}, \nu, H)$, from (22) and (27), since $(P\varphi_h)^+ = (P\varphi_h)^- = 0$ in $\Omega_h \setminus \Omega$.

Also

$$
\begin{aligned}
&D_H{}''(u;\Omega)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) \\
&\quad = D_H{}''(u;\Omega)(P\varphi_h, (P\varphi_h)^+ - (P\varphi_h)^-) \\
&\qquad + D_H{}''(u;\Omega)(\varphi_h - P\varphi_h, (P\varphi_h)^+ - (P\varphi_h)^-) \\
&\qquad\quad + D_H{}''(u;\Omega)(\varphi_h, (\varphi_h^{(+)} - (P\varphi_h)^+) - (\varphi_h^{(-)} - (P\varphi_h)^-)) \\
&\quad \geq \lambda |P\varphi_h|^2_{H^1(\Omega)} + E_2 + E_3
\end{aligned}
$$

from (10). But

$$|E_2|, |E_3| \leq c(1 + \|u\|_{L^\infty}) h^{1/2} |\varphi_h|^2_{H^1(\Omega_h)}$$

from (24) and (23), and (27) respectively.

It follows that

$$
\begin{aligned}
D''_H(J_h u; \Omega_h)(\varphi_h, \varphi_h^{(+)} - \varphi_h^{(-)}) &\geq \lambda |P\varphi_h|^2_{H^1(\Omega)} - ch^{1/2} |\varphi_h|^2_{H^1(\Omega_h)} \\
&\geq \frac{3\lambda}{4} |\varphi_h|^2_{H^1(\Omega_h)}
\end{aligned}
$$

from (24), for $h \leq h_1 = h_1(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \Omega, d, H, \lambda)$.

Lemma 15. *If $v_h \in u_h^0 + X_{h0}$ and $\varphi_h, \psi_h \in X_{h0}$ then*

$$|D''_H(v_h; \Omega_h)(\varphi_h, \psi_h) - D''_H(J_h u; \Omega_h)(\varphi_h, \psi_h)| \leq \frac{\lambda}{4} |\varphi_h|_{H^1(\Omega_h)} |\psi_h|_{H^1(\Omega_h)}$$

provided $|v_h - J_h u|_{H^1(\Omega_h)} \leq \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(H, \lambda)$.

Proof. This follows from

$$
\begin{aligned}
&D''_H(v_h; \Omega_h)(\varphi_h, \psi_h) - D''_H(J_h u; \Omega_h)(\varphi_h, \psi_h) \\
&\qquad = 2HV_h'''(J_h u; \Omega_h)(v_h - J_h u, \varphi_h, \psi_h),
\end{aligned}
$$

(21) and Lemma 11.

Completion of proof of Main Theorem. We use 9 with $\mathcal{X} = u_h^0 + X_{h0}$, $X = X_{h0}$, $Y = X^*_{h0}$, $x_0 = J_h u$, $f = D'_H(\cdot\,; \Omega_h)$. The norm on X_{h0} is $|\cdot|_{H^1(\Omega_h)}$ and on X^*_{h0} is the corresponding dual norm. Note that

$$D'_H(\cdot\,; \Omega_h) : \; u_h^0 + X_{h0} \to X^*_{h0}$$

with derivative

$$D''_H(\cdot\,; \Omega_h) : \; u_h^0 + X_{h0} \to L(X_{h0}, X^*_{h0})$$

using standard identifications.

From Lemma 12

$$\|D'_H(J_h u; \Omega_h)\| \le c_2 h. \tag{29}$$

From Lemma 14, $D''_H(J_h u; \Omega_h)$ is invertible and

$$\left\|[D''_H(J_h u; \Omega_h)]^{-1}\right\| \le \left(\frac{3\lambda}{4}\,|\varphi_h|_{H^1(\Omega_h)} \Big/ |\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h)}\right)^{-1}$$

provided $h \le h_1$. But

$$\begin{aligned}|\varphi_h^{(+)} - \varphi_h^{(-)}|_{H^1(\Omega_h)} &\le |(P\varphi_h)^+ - (P\varphi_h)^-|_{H^1(\Omega_h)} + |\varphi_h^{(+)} - (P\varphi_h)^+|_{H^1(\Omega_h)}\\ &\quad + |\varphi_h^{(-)} - (P\varphi_h)^-|_{H^1(\Omega_h)}\\ &\le (1 + ch^{1/2})|\varphi_h|_{H^1(\Omega_h)}\end{aligned}$$

where $c = c(\nu)$, from (23) and (27). Hence

$$\left\|[D''_H(J_h u; \Omega_h)]^{-1}\right\| \le \left(\frac{\lambda}{2}\right)^{-1} \tag{30}$$

if $h \le h_3$ where $h_3 = h_3(\|u\|_{H^2(\Omega)}, \|u^0\|_{H^2(\Omega)}, \alpha, \Omega, d, H, \lambda)$.

Finally, from Lemma 15

$$\|D''_H(v_h; \Omega_h) - D''_H(J_h u; \Omega_h)\| \le \frac{\lambda}{4} \tag{31}$$

if $|v_h - J_h u|_{H^1(\Omega_h)} \le \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(H, \lambda)$.

Take $\delta = c_2 h$, $\alpha = \lambda/2$, $\beta = \lambda/4$ and $\varepsilon = \varepsilon_1$. Then from (29)–(31) the hypotheses of Lemma 9 are satisfied provided $h \le h_3, \quad c_2 h \le \frac{\lambda}{4}\varepsilon_1$. This establishes the first (uniqueness) part of the main theorem with $\varepsilon_0 = \varepsilon_1$ and $h_0 = h_0(h_3, \varepsilon_1, \lambda, c_2) = h_0(\|u\|_{H^2}, \|u^0\|_{H^2}, \alpha, \nu, H, \lambda)$.

Taking $\delta = c_2 h$, $\alpha = \lambda/2$, $\beta = \lambda/4$ and $\varepsilon = \lambda^{-1} c_0 h$ the hypotheses of Lemma 9 are again satisfied from (29)–(31) provided $h \le h_3, \quad c_2 h \le \frac{1}{4} c_0 h$. This establishes the second ($O(h)$ convergence) part of the main theorem with $h_0 = h_3$ and $c_0 = 4c_2$. □

4 Numerical Results

In Tables 1 and 2 we present the results of test computations for the explicitly known spherical solutions described in Example 2 with $H = 0.5$ and $\Omega = B_1(0)$. Denote by e_h the error between the continuous solution and the discrete solution in the chosen norm. For two successive grids with grid sizes h_1 and h_2 the experimental order of convergence is

$$eoc = \ln \frac{e_{h_1}}{e_{h_2}} \Big/ \ln \frac{h_1}{h_2}.$$

The test computations confirm the order 1 for the $H^1(\Omega)$-norm and additionally show the order 2 for the $L^2(\Omega)$-norm.

Table 1. Small solution, $H = 0.5$

Nodes	Level	h	L^2-error	L^2-eoc	H^1-error	H^1-eoc
9	2	1.0000	1.0020e-1	–	0.2607	–
25	4	0.7368	3.9040e-2	3.09	0.1822	1.17
81	6	0.4203	1.0682e-2	2.31	9.6455e-2	1.13
289	8	0.2219	2.6916e-3	2.16	4.8223e-2	1.09
1089	10	0.1137	6.6871e-4	2.08	2.3909e-2	1.05
4225	12	0.05736	1.6621e-4	2.04	1.1876e-2	1.03
16641	14	0.02893	4.1401e-5	2.02	5.9160e-3	1.01

Table 2. Large solution, $H = 0.5$

Nodes	Level	h	L^2-error	L^2-eoc	H^1-error	H^1-eoc
81	6	0.4203	1.2292	–	6.1915	–
289	8	0.2219	0.4677	1.51	2.9080	1.18
1089	10	0.1137	0.1610	1.60	1.3131	1.19
4225	12	0.05736	0.04707	1.81	0.5870	1.18
16641	14	0.02893	0.01239	1.94	0.2772	1.09

Figures 2 and 3 show computational results with $\Omega = B_1(0)$, $H = 0.5$ and boundary values $u(e^{i\phi}) = (\cos(\phi), \sin(\phi), (2+\sqrt{3})\cos(2\phi) - 0.5\cos(6\phi))$ on a grid with 8192 triangles. For better visibility the resulting surfaces are scaled, but the boundaries of the solution surfaces are the same.

Figure 4 shows a solution for the annular domain $\Omega = \{\, x \mid 1 < |x| < 2 \,\}$ and boundary data which give knotted boundary curves.

Fig. 2. Small solution, $H = 0.5$

Fig. 3. Large solution, $H = 0.5$

Fig. 4. Annulus, $H = 0.5$

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